1-1-2012

Phase Portraits Of An Epidemic model With Incident Rates

Thi D. Doan
Ryerson University

PHASE PORTRAITS OF AN EPIDEMIC MODEL WITH INCIDENT RATES

by

Thi Doan Doan, B.A, York University, 2005

A thesis

presented to Ryerson University

in partial fulfilment of the

requirements for the degree of

Master of Science

in the Program of

Applied Mathematics

Toronto, Ontario, Canada, 2012

AUTHOR'S DECLARATION

I hereby declare that I am the sole author of this dissertation. This is a true copy of the dissertation, including any required final revisions, as accepted by my examiners.

I authorize Ryerson University to lend this dissertation to other institutions or individuals for the purpose of scholarly research.

I further authorize Ryerson University to reproduce this dissertation by photocopying or by other means, in total or in part, at the request of the other institutions or individuals for the purpose of scholarly research.

I understand that my dissertation may be made electronically available to the public.

PHASE PORTRAITS OF AN EPIDEMIC MODEL WITH INCIDENT RATES

Thi Doan Doan

Master of Science, 2012

Applied Mathematics Program

Ryerson University

ABSTRACT

In this thesis, an epidemic model with nonlinear incidence rate is investigated. The ranges of the parameters involved in the model are given under which the equilibria are positive. By carrying out the qualitative behaviour analysis, it is shown the disease free equilibrium can exhibit saddle-nodes, saddle point or stable node depending on the ranges of the parameters. It is shown that the interior equilibria are saddle point, stable or focus nodes. Furthermore, several numerical solution and graphics are given to support the theoretical analysis.

ACKNOWLEDGEMENTS

I want to start by expressing my incredible gratitude to my supervisor, Professor Kunquan Lan. He provides an inspiration to me over previous year, as a graduate student and I have benefited from his encouragement, enthusiasm, and mathematical insight. I appreciate his works and enthusiasm in the topic and how he pushes me beyond my compatibility. I greatly appreciate his efforts to provide me with the financial supports through his NSERC Discovery grants during the scope of the study.

I thank Professor Kunquan Lan for his details feedback on the work that I have done on this thesis, which help create the final version of this thesis. Also, I would like to thank my colleague, Chandra Limbu for helping me with some concepts and feedback of my works. I have enjoyed studying Biomathematics with him during our study with Dr.Lan.

I want to thank the faculty of Applied Mathematics at Reason University for giving me the opportunity to study at Ryerson and the financial assistance they have given me. I want to thank the professors at Ryerson who early in my graduate had taught me several courses of mathematics including technique and principle for Applied mathematics. Also, I would be remiss not to mention the many graduate students I had the good fortune of befriending at Ryerson and all who had an enormous impact on me as a graduate student.

I would like to thank the members of examining committee, Professor Dr. K.Q Lan, Dr. Isaac Woungang, Dr. Silvana Illie, and the Chair Professor Dr. Pablo Olivares for their times and commitments to review my thesis.

Last but not least, I would like to thank my parents for their unconditional love and supports throughout the years.

DEDICATION

I would like to dedicate my work to my beloved daughter, Jade Lang. You are the greatest joy of my life.

Contents

List of Figures

List of Abbreviation

I	The infected rate
R	The recovery rate or the removed rate
S	The susceptible rate
d	The natural death rate
k	The probability of transmission per contact
kI^p	The measure of the infection force of the disease
a	The inhibition effect
ν	The rate of removed individual who lose immunity and return to susceptible class
r	The recovery rate of the infective individuals
β	The transmission rate
KMK	The Kermack Mckendrick model
$\dfrac{1}{(1+aI^p)}$	The inhibition effect either from the behavioural change of the susceptible individuals as the number of the susceptible individuals increases or from the crowded infective individuals

Preface

This thesis is a record of part of the research carried out by the author during the academic years 2010-1012. It is submitted according to the regulations for the degree of Master of Science in Ryerson University.

Almost all of the results of this thesis are the original work of the author with the exception of several results specifically mentioned in the text and attributed there to the authors concerned.

Chapter 1 and 2 contain preliminary materials, while, chapters 3,4 and 5 consist of new results.

Chapter 1

Introduction

Each year, more than one million people die of communicable diseases [21]. For example, during the Bubonic Plague, also known as the Black Death, an epidemic in Europe first occurring between 1348 to 1350, about thirty to sixty percent of the European population were wiped out. The Bubonic Plague affected Europe again in the early 17th and 18th century, leaving some regions devastated. Approximately one third of the European population perished during that time period [22]. A recent epidemic caused by severe acute respiratory syndrome (SARS), in 2003 had affected more than five thousand people and had taken 750 lives [1]. This serious form of pneumonia is caused by a virus and was first identified in Asia, spreading quickly throughout the world. Another example is the H1N1 Influenza pandemic that occurred in 2009 [21]. The disease is a combination of bird and pig influenza virus, known as **swine flu**. The virus is able to pass from animal to human and between humans. By the time the pandemic had slowed down in 2010, there was a report of 18000 deaths [21]. These diseases can be treatable or preventable [19]. Public health authorities are concerned with a sudden outbreak of disease and an endemic situation, of which the disease is always present.

1.1 Problems

The questions that are to be addressed are [12]:

1. How many individuals will be affected, and if so, how many will require treatment?

2. How many individuals are needed to be isolated before the disease will die out?

3. What is the number of vaccines needed to reduce the spread of disease?

In order to address the above issues, the public authorities and researchers must conduct time consuming and expensive experiments.

Many mathematicians are following the footsteps of pioneers to come up with a more realistic model so that they can predict or answer some basic questions regarding to the outcome of disease. After Kermack Mckendrick studied the plaque spread in London from 1665-1666, he introduced the first epidemic model, SIR model, which considers a fixed population with three compartments described in Chapter 3: Susceptible, Infected, and Recovery. This model is used to predict the outcome of the disease. However, it is beneficial to know how to control an epidemic before it happens. In mathematical disease modelling, it is only a tool which has been used to study the mechanism of the disease spreading. Furthermore, it is used to predict the future course of an outbreak and to evaluate strategies to control an epidemic [16, 12].

In epidemic modelling, there are two types of models, simple and detailed. There are drawbacks to both types. Simple models do not allow for a variety of situations, including vaccinations, quarantine, or delayed onset of the disease. Detailed models are usually designed for specific circumstances including short-term prediction with situations unusable in simple models, though they are impossible to solve analytically and are only used for theoretical purposes[12]. For instance, the disease model is always associated with the threshold behaviour which is used to determine whether an epidemic occurs or dies out. In [13, 14, 27], the authors use a reproduction number, known as R_0 to describe an epidemic. If $R_0 > 1$, there will be an epidemic; if $R_0 < 1$, the epidemic will die out; and if $R_0 = 1$, there will be an endemic, which mean the disease will re-occur.

1.2 Objectives

In this thesis, we do not intend to address the problems discussed above. The detailed models with the modified incident rates will be used to analyze an epidemic for theoretical purposes. The primary objectives of this thesis are:

1. To modify the non-linear incident rate and study the dynamical behaviours of the reduced IR system that is the equation of (3.4).

2. To find the number of equilibria and study the qualitative behaviour at each equilibrium.

3. To determine the value for the parameters α, β, γ and δ to clarify the stability of the disease.

In case one, we will modify the incident rate in the form of $\dfrac{kI^pS^q}{1+aI^p}$, where $p = 1$ and $q = 2$. The incident rate is proposed by Liu et al. and is used by several authors [18, 23, 24]. More details and problems arise with the incident rate will be explained in Chapter 3. For case two, the number of equilibria are associated with the change in incident rate. After finding the number of equilibria, the qualitative behaviour will be studied. Last part, we will restrict the value of the parameters α, β, γ and δ in our model of interest so that we can determine the stability of the disease.

1.3 Methodology

We are interested in the study of the epidemic model with non-linear incident rate . First, we will change the variable of the parameters, α, β, δ and γ, so that our model of interest will become a non-dimensional system of equation (3.5). Second, we will use the techniques such as linearization and nullcline in Chapter 2 to solve the two dimensional system and to determine the number of equilibria. The last part of the thesis is to analyse the stability of the equilibrium by using Theorem 2.1.4, Lemma 2.1.5 and Lemma 2.3. We will conclude our thesis with numerical simulation for the phase portraits of each equilibrium point by using the software called Maple13 [4].

1.4 Thesis Organization

This thesis is devoted to the study of the dynamical behaviour of a particular class epidemic SIRS model with non-linear incidence rate, which we will introduce the model in Chapter 3. Our goals are to find the total number of equilibria, to investigate the phase portraits near the equilibria and to understand whether the disease will spread or persist for our model of interest.

In Chapter 1, there will a brief introduction to the epidemiology, problems and the explanations of the thesis objectives.

In Chapter 2, we will study the tools and terminology to analyze the results of our model of interest. Also, we will study the background for our model of interest and how it evolves. An illustration of our model will be introduced in this chapter.

Chapter 3 and 4 contain the results and Chapter 5 contains the conclusion of our model of interest. We will use numerical simulation by using the software calls Maple13 to illustrate our results.

Chapter 2

Non-Linear Differential Equation′s Theories and Techniques

2.1 Introduction

In this chapter, we will present the mathematical theories and techniques that are useful to study non-linear differential equations. It will be useful to interpret the biological meaning of our epidemic model. The techniques are used to study the behaviour of the dynamics of a system as the parameters change such as limited cycle and stability. Since our model is a non-trivial non-linear differential equation , it is impossible to find explicit solutions. Numerical simulation or study of the stability of the system at the equilibrium is the only way to draw a conclusion to our model regardless whether we know the solution or not. In order to understand and to analyze our model of interest, we have to study the equilibrium solution of non-linear system. Before we can do that, we must understand what is the definition of the equilibrium . In [16] Allen, stated that "the equilibrium solution is biologically interesting because they represent resting states or stationary states of the system". For example, if the zero is an equilibrium point, then it can represent the disease-free states of an epidemic model.

2.1.1 Definition of Autonomous Differential Equation

Our epidemic model of interest is an autonomous differential equation , that means the functions $f_1(x, y)$ and $g_1(x, y)$ do not depend on the independent variable, which we denote by t for time. To interpret the biological meaning of autonomous model, the infected , the removed , and the susceptible class do not depend explicitly on time; this means the infected, susceptible and removed class should yield the same outcome regardless when the disease started.

Our model deals with non-linear incident rate . As stated in Liu et Al [27], "the incidence rates are the rates of new infections involved in SIRS model by giving reasonable qualitative description of dynamic of the diseases".

We consider the following two dimensional planar systems:

$$\begin{cases} \dfrac{dx}{dt} = f_1(x, y), \\ \dfrac{dy}{dt} = g_1(x, y), \end{cases} \tag{2.1}$$

where $f_1, g_1 : X \to \mathbb{R}$ are functions having continuous first partial derivatives and X is an open subset in R^2.

2.1.2 Equilibrium

Since we restrict all the parameters to be positive, the equilibrium point must be positive to have biological meaning. If it is negative, then the solution tends to extinction (the disease dies out).

Definition 2.1.1. [8] If $(\bar{x}, \bar{y})$ *is an equilibrium point or fixed point of* (2.1) if it satisfies $f_1(\bar{x}, \bar{y}) = 0$ and $g_1(\bar{x}, \bar{y}) = 0$. An equilibrium point $(\bar{x}, \bar{y})$ of (2.1) is said to be *positive* if $(\bar{x}, \bar{y}) \in P$ where

$$P = \{(\bar{x}, \bar{y}) \in \mathbb{R}^2 : \bar{x} \geq 0 \quad \text{and } \bar{y} \geq 0\},$$

and to be an *interior equilibrium point* if $\bar{x} > 0$ and $\bar{y} > 0$.

2.1.3 Nullcline

The phase plane portrait is the direction field that can be used as "visual" aid in sketching diagram [5, 16] . To analyze the phase plane, all we need to know is whether the flow is up or down on the x-nullcline , the flow is left or right for the y-nullcline.

Definition 2.1.2. [16] The *x-zero isocline* or *nullcline* for system (2.1) is the set of all points in the (x, y) plane satisfying $f_1(x, y) = 0$. The *y-zero isocline* or *nullcline* is the set of all points satisfying $g_1(x, y) = 0$.

2.1.4 Phase Plane Analysis

As mention in our introduction, we need to understand the theory behind the dynamic behaviours of a system of two differential equations. We need to study the local stability through phase plane analysis near the fixed point. One way to do that is to linearize a non-linear system about an equilibrium.

Definition 2.1.3. [6, 16] *Linearization* is a method to determine the local stability of an equilibrium of a system of non-linear differential equation . It is a technique that has been used for studying linear system to analyze the behaviour of a non-linear function near the fixed point.

Let the system of (2.1) be a non-linear system . We will expand the function f and g by using Taylor formula. Now, we are going to apply a small perturbation with $u = x - \bar{x}$ and $v = y - \bar{y}$. Then

$$\frac{du}{dt} = f(\bar{x}, \bar{y}) + f_x(\bar{x}, \bar{y})u + f_y(\bar{x}, \bar{y})v + f_{xx}(\bar{x}, \bar{y})\frac{u^2}{2} + ...,$$
$$\frac{dv}{dt} = g(\bar{x}, \bar{y}) + g_x(\bar{x}, \bar{y})u + g_y(\bar{x}, \bar{y})v + g_{xx}(\bar{x}, \bar{y})\frac{u^2}{2} + ...,$$

where $f_x(\bar{x}, \bar{y}) = \frac{\partial f(x, y)}{\partial x}|_{x=\bar{x}, y=\bar{y}}$ and $g_x(\bar{x}, \bar{y}) = \frac{\partial g(x, y)}{\partial x}|_{x=\bar{x}, y=\bar{y}}$ and so on. We will neglect the partial derivatives of the terms of order greater or equal to two. Recall from definition 2.1.1, the equilibria of the system (2.1) are solution $(\bar{x}, \bar{y})$ that satisfy $f(\bar{x}, \bar{y}) = 0$ and $g(\bar{x}, \bar{y}) = 0$.

The system linearized about the equilibrium $(\bar{x}, \bar{y})$ is [16]

$$\frac{dZ}{dt} = JZ,$$

where $Z = (u, v)^T$ and J is the *Jacobian matrix* evaluated at the equilibrium

.

Lets recall some results on phase portraits of planar systems near equilibria in the qualitative theory [7, 8, 9]. Let us denote by $A(x, y)$ the Jacobian matrix of f_1 and g_1 at (x, y), that is,

$$A(x,y) = \begin{pmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial g_1}{\partial x} & \dfrac{\partial g_1}{\partial y} \end{pmatrix} \tag{2.2}$$

and by $|A(x, y)|$ and $tr(A(x, y))$ the determinant and the trace of $A(x, y)$, respectively.

Then the characteristic polynomial of $A(x, y)$ is:

$$\lambda^2 - tr(A(x,y)) + det(A(x,y))$$

It is well known that the solutions of a planar system near its equilibria (x, y) can be studied by the eigenvalues of $A(x, y)$, which can be determined by $|A(x, y)|$ and $tr(A(x, y))$.

Theorem 2.1.4. *[16] Assume the first - order partial derivatives of f and g are continuous in some open set containing the equilibrium (x_0, y_0) of system* (2.1).

(i) Then the equilibrium is locally asymptotically stable if

$tr(A(x, y)) < 0$ and $det(A(x, y)) > 0$.

(ii) The equilibrium is unstable if $tr(A(x, y)) > 0$ or $det(A(x, y)) < 0$,

where $A(x, y)$ is the Jacobian matrix evaluated at the equilibrium .

The classification schemes for the non-linear system case should be the same for the linear case after we linearized the non-linear system. Subsequently Linearization only gives us the approximation of the non-linear system, then it mights behave differently from the linear system. For general proof see [6, 16]. Here are the three cases in which a non-linear system behave differently from the linear system [16]:

1. $det(A(x,y)) = 0$. There is at least one zero eigenvalue . If there is an isolated equilibrium , it can be a node, spiral or saddle.

2. $tr(A(x,y)) = 0$ and $det(A(x,y)) > 0$. The eigenvalues are purely imaginary. The equilibrium may be a center or spiral .

3. $tr(A(x,y))^2 = 4(det(A(x,y))$. This represents the borderline between complex and real eigenvalues. Thus, the equilibrium can be a node or a spiral.

The Figure 2.1 illustrates the classification schemes of a non-linear system that has been linearized to linear system . The stability diagram in the (T, Δ) plane, where $T = tr(A(x,y))$ and $\Delta = det(A(x,y))$, and $\Delta = T$. The line curves represent the three.

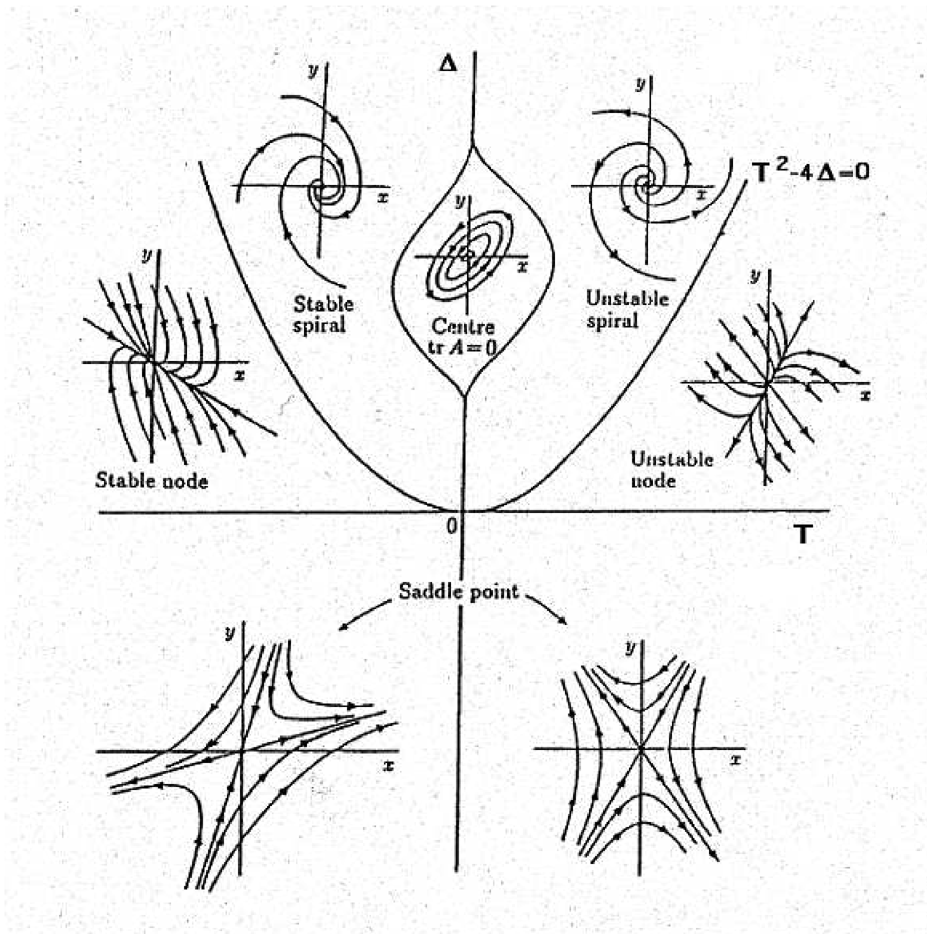


Figure 2.1: [11] The stability diagram of non-linear system.

2.1.5 Classification Scheme of Stability

The following classification schemes can be found in [7, 8, 9] and have been used in [7, 8, 9].

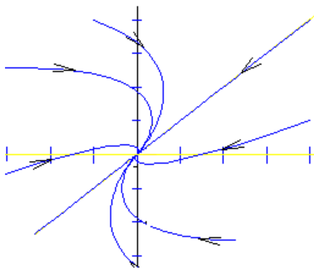
Lemma 2.1.5. *[8] If (x_0, y_0) is an equilibrium of (2.1), then the following assertions hold.*
(i) If $|A(x_0, y_0)| < 0$, then (x_0, y_0) is a saddle of (2.1).
(ii) If $|A(x_0, y_0)| > 0$, $(tr(A(x_0, y_0)))^2 - 4|A(x_0, y_0)| \geq 0$ and $(A(x_0, y_0)) \neq 0$, then (x_0, y_0) is a node of (2.1); it is stable if $tr(A(x_0, y_0)) < 0$ and unstable if $tr(A(x_0, y_0)) > 0$.
(iii) If $|A(x_0, y_0)| > 0$, $(tr(A(x_0, y_0)))^2 - 4|A(x_0, y_0)| < 0$ and $tr(A(x_0, y_0)) \neq 0$, then (x_0, y_0) is a focus of (2.1); it is stable if $tr(A(x_0, y_0)) < 0$ and unstable if $tr(A(x_0, y_0)) > 0$.

Lemma 2.1.6. *[8] Let (x_0, y_0) be an equilibrium of (2.1). Assume that $|A(x_0, y_0)| = 0$, $tr(A(x_0, y_0)) \neq 0$ and (2.1) is equivalent to the following system*

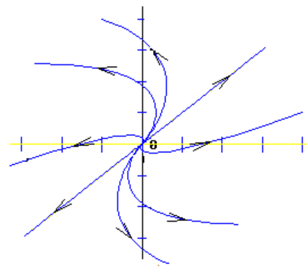
$$\begin{cases} \dot{u} = p(u, v), \\ \dot{v} = \varrho v + q(u, v) \end{cases} \tag{2.3}$$

with an isolated equilibrium point $(0, 0)$, where $\varrho \neq 0$,
$p(u, v) = \sum_{i+j=2,i,j\geq 0}^{\infty} a_{ij} u^i v^j$ and $q(u, v) = \sum_{i+j=2,i,j\geq 0}^{\infty} b_{ij} u^i v^j$ are convergent power series. If $a_{20} \neq 0$, then (x_0, y_0) is a saddle-node of (2.1).

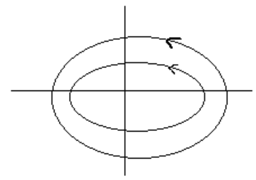
The graphs and explanations of the Lemma (2.1.5) and (2.1.6) are provided below:



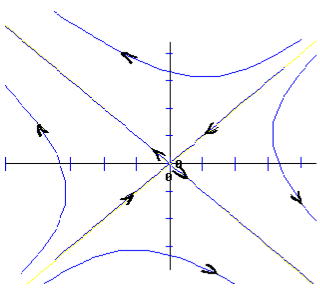
(a) Stable node – the solution will flow into the origin or the solution will approach the equilibrium regardless of the starting point.



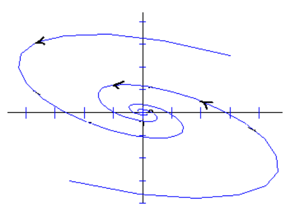
(b) Unstable node -the solution will not converge to the equilibrium, unless it starts at the equilibrium.



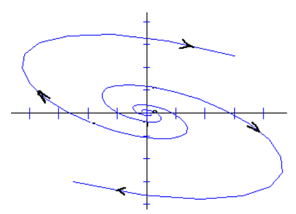
(c) Center -the solution will neither approach nor move away from the equilibrium



(d) Saddle point -the solution will not converge to the equilibrium



(e) Stable Spiral -the solution starts from any point other than the equilibrium, it spirals into the equilibrium .



(f) Unstable Spiral -the solution starts from the fixed point and then spirals away from the equilibrium.

Figure 2.2: Phase portrait diagram

Chapter 3

Introduction to Epidemic Model

3.1 Introduction

The first section of this chapter is devoted to the background of the Epidemic model . This will include the mechanism of the model and its first pioneer that came up with the model. Furthermore, the history of the SIRS models will be introduced and its limitations that arise with the model. A flow chart of SIRS model will be included in this section. In the second part of this chapter, we will present our model of interest. The model is an expansion of the work from [18, 24]. The last part of the chapter is devoted to find the number of equilibria . We will use the theories and techniques from Chapter 2 to find the number of equilibria and will also determine the Jacobian matrix for our model of interest.

3.2 Background of Epidemic Model

Between 1665-1666, when Kermack Mckendrick studied the Bubonic plaque in London, he introduced a new epidemic model known as an SIR model, which is also commonly referred to as the Kermack McKendrick model (KMK) [13]. The epidemic model in the form of SIRS model is an extension of the classical deterministic epidemic of SIR models. The SIRS model is used to model many infectious diseases such as influenza, measles and mumps. The model is divided into three subclasses: the susceptible class, the infective class and the recovery or removed class. The susceptible class represents the individuals who are capable of contracting the disease and becoming infected, while the infective class is the class of those individuals who are capable of transmitting the diseases. The removed class represents those who have had the disease, recovered from it and will then re-enter into the susceptible class.

There are few drawbacks with the KMK model . It does not capture a realistic situation because in the model, the interaction term is a linearly increasing function of the number of infected persons. The KMK only describes the true behaviour for a small number of infective. Moreover, Capasso and Serio [25] show that this is always true as the number of contacts of a susceptible per unit time will not increase linearly with infective. Capasso and Serio [25] generalize the incidence rate in the form of $g(I)$ into the SIRS model when they observe the cholera epidemic spread of Bari in 1973 [25]. The saturate incidence rate allows the crowding effect of infective individual when the susceptible and infective individual may saturate at high levels.

However, the model requires that $g'(0)$ must be positive and finite. Furthermore, Wilson and Worcester introduce the first general incidence rate with S^p in 1945 but the incidence rate does not fit the data well. In 1969, Severo introduces a more general form of incidence rate in the form of KI^pS^q where $q < 1$. The problem with Severo′s model however, is that it does not investigate the behaviour of the incidence rate in full. In 1975, Bailey introduces bilinear incidence rate into the SIRS model, and it is in the form of βIS where β is the transmission rate. The problem with the bilinear incidence rate in Baileys model is that it contains trivial equilibrium where $I = R = 0$, which means there is no disease present. Therefore, the effect of behavioural changes has been incorporated by Liu et al [14, 23] through the use of a non-linear incidence rate $\dfrac{KI^lS}{1+\alpha I^h}$ with $k, l, \alpha, h > 0$. The incidence rate of the form $\dfrac{KI^lS}{1+\alpha I^h}$ is more reasonable then βI^pS^q because it includes the change in behavioural and crowding effect of the infective individual and prevents the unboundedness of the contact rate by choosing the suitable parameters [24]. The flow chart of Figure 3.1 below illustrates the cycle of SIRS models.

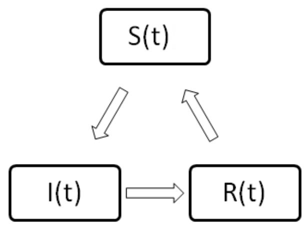


Figure 3.1: Illustrates the flow chart of SIRS model, where $S(t)$ represents the susceptible rate, $I(t)$ corresponds to the infected rate, while $R(t)$ denotes the recovery or the removed rate. In the SIRS model, the recovered individual may lose the immunity and re-enter the susceptible state.

3.3 Previous results

Our result is different from the result in [18, 24]. In [18], the authors do not specify the exact value for the number equilibria. The equilibria are determined by the basic reproductive number, known as R_0. Moreover, the bifurcation and qualitative analyses are generalized by the change of parameters and R_0.

On the other hand, in [24], after they modify the incident rate in the form of $\dfrac{KI^lS}{1+\alpha I^h}$, where $l = 2$, they study the global qualitative and bifurcation analyses. The authors show that the system had at least two limit cycles and the system undergoes a Bogdanov-Takens bifurcation at the degenerate equilibrium with the suitable conditions for the parameters. However, for our SIRS model, the system does not undergoes a Bogdanov-Takens bifurcation nor it has any limit cycles. This mean, the existence of a persistence region of the disease does not occur in our model of interest. In our system, the disease either persistence-outbreak or the disease is well controlled with the suitable condition for the parameters α, β, δ and γ. For more details of our result, see discussion in Chapter 5.

3.4 Preliminary

In this section, we will extend the previous study from [18, 24] with the epidemic model of great interest in the form of:

$$\begin{cases} \dot{S} = B - dS - \left(\dfrac{kI^p S^q}{1 + aI^p} \right) + \nu R, \\ \dot{I} = \left(\dfrac{kI^p S^q}{1 + aI^p} \right) - (d + r)I, \\ \dot{R} = rI - (d + \nu)R, \end{cases} \tag{3.1}$$

with the non-linear incidence rate per infective individual given by [18]

$$H(I, S) := \frac{I^p S^q}{1 + aI^p}, \tag{3.2}$$

where B is the recruitment rate of population including newborns and immigrants, d is the natural death rate, S, I and R represent the susceptible populations who can catch the disease, infected populations who have the disease and can transmit it to others, and the removed populations who have either had the disease or are dead, are recovered , are permanently immune or are isolated until recovered, respectively, k is the probability of transmission per contact, kI^p measures the infection force of the disease and $1/(1 + aI^p)$

with $a \geq 0$ denotes the inhibition effect either from the behavioural change of the susceptible individuals as the number of the susceptible individuals increases or from the crowded infective individuals, ν is the rate of removed individuals who lose immunity and return to susceptible class and r is the recovery rate of the infective individuals.

There have been studies on dynamical behaviour of (3.1) with some specific numbers for p and q under the standard assumption that:

$$S(t) + I(t) + R(t) = N_\infty \text{ for } t \geq 0 \tag{3.3}$$

whereN_∞ represents the population in the equilibrium in the absence of the disease. The equation (3.3) has been used in [18, 8, 27], when the population size reaches its limit value N_∞. We will consider the case for (3.1) with $q = 2$ and $p = 1$ under the same assumption as (3.3) and study the dynamical behaviours of the reduced IR system that is the last two equations of (3.1), by replacing $S(t)$ replaced by $N_\infty - I(t) - R(t)$. The reduced IR system is the projection of (3.1) with $q = 2$ and $p = 1$ to the IR planar coordinate system .

3.5 Positive Equilibria of (x_1, y_1) and (x_2, y_2)

In this section, we derive an equivalent system of the reduced IR system of (3.1) with $q = 2$ and $p = 1$, under the assumption of $S(t) = N_\infty - I(t) - R(t)$ and we replace the susceptible populations $S(t)$ in the first equation of (3.1).

Thus we obtain the reduced IR system:

$$\begin{cases} \dot{I} = \dfrac{kI(N_\infty - I - R)^2}{1+aI} - (d+r)I, \\ \dot{R} = rI - (d+\nu)R. \end{cases} \tag{3.4}$$

By rescalling (3.4), we let $x = I/N_\infty$, $y = R/N_\infty$ and $\check{t} = (d+\nu)t$, and then apply the Chain rule to the following equations:

$$\begin{aligned} \frac{dI}{dt} &= N_\infty \frac{dx}{d\check{t}} \frac{d\check{t}}{dt} = N_\infty (d+\nu) \frac{dx}{d\check{t}} \\ \frac{dR}{dt} &= N_\infty \frac{dy}{d\check{t}} \frac{d\check{t}}{dt} = N_\infty (d+\nu) \frac{dy}{d\check{t}} \end{aligned}$$

Now, we substitute the values for $\dfrac{dI}{dt}$, $\dfrac{dR}{dt}$, I and R into (3.4).

$$\begin{aligned} N_\infty (d+\nu) \frac{dx}{d\check{t}} &= \frac{k(xN_\infty)(N_\infty - xN_\infty - yN_\infty)^2}{1 + axN_\infty} - (d+r)xN_\infty \\ \frac{dx}{d\check{t}} &= \frac{k(xN_\infty^3)(1-x-y)^2}{(1+axN_\infty)(N_\infty(d+\nu))} - \frac{(d+r)xN_\infty}{N_\infty(d+\nu)} \\ \frac{dx}{d\check{t}} &= \frac{\beta x(1-x-y)^2}{1+\alpha x} - \gamma \end{aligned}$$

$$\begin{aligned} N_\infty (d+\nu) \frac{dy}{d\check{t}} &= rxN_\infty + (d+\nu)yN_\infty \\ \frac{dy}{d\check{t}} &= \frac{rxN_\infty}{N_\infty(d+\nu)} + \frac{(d+\nu)yN_\infty}{N_\infty(d+\nu)} \\ \frac{dy}{d\check{t}} &= \delta x - y \end{aligned}$$

The system (3.4) will become the following non-dimensional system:

$$\begin{cases} \dot{x} = \dfrac{\beta x(1-x-y)^2}{1+\alpha x} - \gamma x := f(x,y), \\ \dot{y} = \delta x - y := g(x,y), \end{cases} \tag{3.5}$$

Where $\alpha = aN_\infty, \beta = \dfrac{kN_\infty^2}{d+\nu}, \delta = \dfrac{r}{(d+\nu)}$, and $\gamma = \dfrac{d+r}{d+\nu}$. However, the parameters α, β, δ, and γ are having the similar biological meanings as a, k, d and ν. From now on, we study the dynamical behaviours of (3.5). (x, y) is an equilibrium point of (3.5) if only if (x, y) satisfies the following system:

$$\begin{cases} \dfrac{\beta(1-x-y)^2}{1+\alpha x} - \gamma = 0, \\ y = \delta x \text{ and } x \neq 0. \end{cases} \tag{3.6}$$

The following result gives solutions of (3.6).

Notation :

$$\Delta = \alpha^2\gamma^2 + 4\beta\gamma(1+\delta)^2 + 4\beta\alpha\gamma(1+\delta) \tag{3.7}$$

$$x_1 = \frac{2\beta(1+\delta) + \alpha\gamma - \sqrt{\Delta}}{2\beta(1+\delta)^2} \text{ and } y_1 = \delta x_1 \tag{3.8}$$

$$x_2 = \frac{2\beta(1+\delta) + \alpha\gamma + \sqrt{\Delta}}{2\beta(1+\delta)^2} \text{ and } y_1 = \delta x_2 \tag{3.9}$$

Lemma 3.5.1. *Assume that* $\alpha, \delta > 0$. *Then*
$\gamma < \beta$ *if and only if* $2\beta(1+\delta) + \alpha\gamma > \sqrt{\Delta}$

Lemma 3.5.2. *Assume that* $\alpha, \delta > 0$. *Then the following assertions hold.*

(1) If $0 < \beta \leq \gamma$, *then* (3.5) *has one solution* (x_2, y_2), *where* $0 < x_2$ *is given in* (3.9).

(2) If $0 < \gamma < \beta$, *then* (3.5) *has two solutions* (x_1, y_1) *and* (x_2, y_2), *where* $0 < x_1 < x_2$ *is given in* (3.8) *and* (3.9).

Proof. Substituting $y = \delta x$ into the first equation of (3.5), we obtain

$$x^2 - \frac{2\beta(1+\delta)+\alpha\gamma}{\beta(1+\delta)^2}x + \frac{(\beta-\gamma)}{\beta(1+\delta)^2} = 0 \tag{3.10}$$

Therefore,

$$\left[x - \frac{2\beta(1+\delta)+\alpha\gamma}{2\beta(1+\delta)^2}\right]^2 = \frac{\Delta}{4\beta^2(1+\delta)^4} \tag{3.11}$$

where $\Delta = (2\beta(1+\delta)+\alpha\gamma)^2 - 4\beta(1+\delta)^2(\beta-\gamma)$. By computation, we can see that

$$\begin{aligned} \Delta &= 4\beta^2(1+\delta)^2 + 4\beta\alpha\gamma(1+\delta) + \alpha^2\gamma^2 - 4\beta^2(1+\delta)^2 + 4\beta\gamma(1+\delta) \\ &= \alpha^2\gamma^2 + 4\beta\gamma(1+\delta)^2 + 4\beta\alpha\gamma(1+\delta) \end{aligned}$$

Therefore, it follows from (3.11) that $\Delta > 0$. To see how many positive solutions x and y contain, we need to set

$$2\beta(1+\delta)+\alpha\gamma > \sqrt{\alpha\gamma + 4\beta\gamma(1+\delta)^2 + 4\beta\gamma\alpha(1+\delta)} \tag{3.12}$$

$$\begin{aligned} 2\beta(1+\delta)+\alpha\gamma &> \sqrt{\alpha\gamma + 4\beta\gamma(1+\delta)^2 + 4\beta\gamma\alpha(1+\delta)} \\ 4\beta^2(1+\delta^2) + 4\beta\alpha\gamma(1+\delta) + \alpha^2\gamma^2 &> \alpha^2\gamma^2 + 4\beta\gamma(1+\delta)^2 + 4\beta\gamma\alpha(1+\delta) \\ 4\beta^2(1+\delta)^2 &> 4\beta\gamma(1+\delta)^2 \\ \beta &> \gamma \end{aligned}$$

Thus Lemma (3.5.1) holds.

If $\beta < \gamma$, then $x_1 < 0 < x_2$. Since x_1 is negative and only $x_2 > 0$, then the result for Lemma (3.5.2) (1) holds.

When $\beta > \gamma$, then $x_1 > 0$ and $x_1 > x_2$. Thus the result holds for Lemma (3.5.2) (2). $\square$

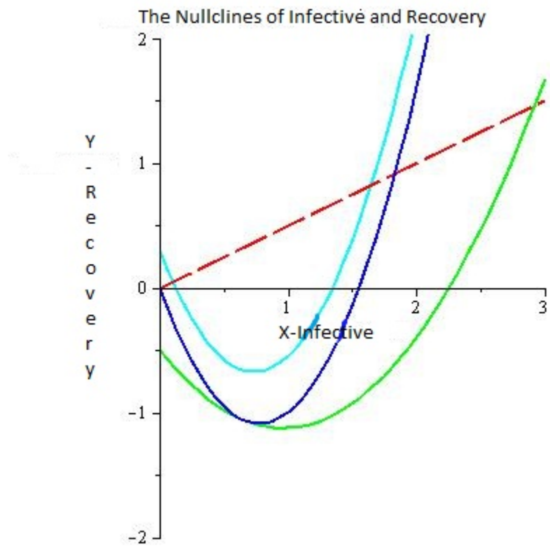


Figure 3.2: Illustrates the results of Theorem 3.5.3 and is obtained from (3.6) with $f(x, y) = 0$ and $g(x, y) = 0$. By letting $\delta = \alpha = 0.5$ and changing the value for β and γ respectively. The solid cyan line is obtained by letting $\beta > \gamma$. The solid blue line is the result of $\beta = \gamma$, while, the green solid line is obtained by using $\beta < \gamma$.

Theorem 3.5.3. *Assume that* $\alpha > 0$, *and* $\delta > 0$, *then the following assertions hold.*

(i) If $0 < \beta \leq \gamma$, *then the equation of* (3.5) *has two equilibria* $(0, 0)$, *and* (x_2, y_2) *in* P.

(ii) If $0 < \gamma < \beta$, *then the equation of* (3.5) *has three equilibria* $(0, 0), (x_1, y_1)$ *and* (x_2, y_2) *in* P.

3.6 The Jacobian Matrix of (x^*, y^*)

Before we can study the stability of the phase portraits of (3.5), we need to define the Jacobian matrix for (3.5). The Jacobian matrix is derived from the partial derivative of (3.5)

Notation:

$$\begin{aligned} r_1(\delta, \beta, \delta, \gamma) &= \frac{(-2\beta x^*)(1+\alpha x^*)(1-x^*-y^*) + \beta(1-x^*-y^*)^2}{(1+\alpha x^*)^2} - \gamma \\ r_2(\beta, \alpha) &= \frac{-2\beta x^*(1-x^*-y^*)}{1+\alpha x^*} \end{aligned}$$

Lemma 3.6.1. *Assume that $\alpha, \delta, \gamma > 0$ and $\beta > 0$, then the Jacobian Matrix for (3.5) is:*

$$A(x^*, y^*) = \begin{pmatrix} r_1(\delta, \beta, \delta) & r_2(\beta, \alpha) \\ \delta & -1 \end{pmatrix} \tag{3.13}$$

Proof. With the combination of equations (3.5) and (2.1), we obtain the partial derivatives

$$\begin{aligned} \frac{df}{dx} &= \frac{(1+\alpha x)\left[2\beta x(1-x^*-y^*)(-1) + \beta(1-x^*-y^*)^2\right] - \alpha\left[\beta x(1-x^*-y^*)^2\right]}{(1+\alpha x)^2} - \gamma \\ &= \frac{(-2\beta x^*)(1+\alpha x^*)(1-x^*-y^*) + \beta(1-x^*-y^*)^2}{(1+\alpha x^*)^2} - \gamma \\ &= \frac{(-2\beta x^*)(1-x^*-y^*) + \gamma}{(1+\alpha x^*)} - \gamma = r_1(\delta, \beta, \delta, \gamma) \\ \frac{df}{dy} &= \frac{2\beta x^*(1-x^*-y^*)(-1)}{1+\alpha x^*} = \frac{-2\beta x^*(1-x^*-y^*)}{1+\alpha x^*} = r_2(\beta, \alpha) \\ \frac{dg}{dx} &= \delta \text{ and } \frac{dg}{dy} = -1. \end{aligned}$$

$\square$

Chapter 4

Qualitative Behaviour Analysis

4.1 Introduction

As mention earlier in Chapter 2, SIRS model is a non-linear differential equation and it is impossible to find explicit solutions. Thus, we can only study the dynamical behaviour of the system by analysing the phase portraits near the equilibrium .

In this chapter, we will use Theorem 2.1.4 and Lemma (2.1.5) to analyze the qualitative behaviour of the system (3.5). First, we will investigate the disease-free equilibrium $E_0(0, 0)$. Second, we will analyze the fixed point for (x_1, y_1) and (x_2, y_2) and last, we will use graphical approach to conclude our chapter.

4.2 Phase portraits at the disease-free equilibrium

We use the theory mentioned above to study the phase portraits of (3.5) near the equilibria. First, we need to start with the disease-free equilibrium $E_0(0,0)$.

Theorem 4.2.1. *(i) If* $\alpha > 0, \delta > 0, 0 < \gamma < \beta$, *then* $(0,0)$ *is a saddle of* (3.5)

(ii) If $\alpha > 0, \delta > 0, \beta = \gamma$, *then* $(0,0)$ *is a saddle node of* (3.5)

(iii) If $\alpha > 0, \delta > 0, 0 < \beta < \gamma$, *then* $(0,0)$ *is a stable node of* (3.5)

Proof. Let $(0,0)$ be a solution of (3.5), then the Jacobian matrix of (3.5) is

$$A(0,0) = \begin{pmatrix} \beta - \gamma & 0 \\ \delta & -1 \end{pmatrix}$$

$$\begin{aligned} \text{And determinant is: } |A(0,0)| &= (\beta-\gamma)(-1) - 0 = \gamma - \beta \\ \text{And the trace is: } tr(A(0,0)) &= \beta - \gamma - 1 \end{aligned}$$

(a) If $\beta > \gamma$ then the condition applies to the following situations:
$|A(0,0)| = (\gamma - \beta) < 0$, thus (i) holds.
(b) If $\beta = \gamma$, the determinant of $A(0,0)$ will be $|A(0,0)| = (\beta - \beta) = 0$, therefore we will use Lemma 2.1.6 to prove (ii).
(c) if $\beta < \gamma$ then the determinant of $A(0,0)$ is always positive and the trace of $A(0,0)$ is always negative. The $|A(0,0)| = (\gamma - \beta) > 0$ and $tr(A(0,0)) = (\beta - \gamma - 1) < 0$, thus (iii) holds. □

4.2.1 Saddle node at the Origin

Notation :

$$a_0 = \beta - \gamma,\ a_1 = -2\beta,\ a_2 = \beta\ a_3 = -\beta(\alpha + 2)$$

$$a_4 = 2\beta(\alpha + 1),\ a_5 = -\beta\alpha,\ b_0 = \delta,\ b_1 = -1$$

Lemma 4.2.2. *Let* $(0, 0)$ *be an equilibrium of* (3.5)*. Then the following assertions hold.*

(1) Since $E_0(0, 0)$ *is at the origin, the equation* (3.5) *can be changed into the following system*

$$\begin{cases} \dot{x} = a_0x + a_1xy + a_2xy^2 + a_3x^2 + a_4x^2y + a_5x^2y^2 + O_3(x, y) := f_1(x, y), \\ \dot{y} = b_0x + b_1y := g_1(x, y) \end{cases} \tag{4.1}$$

where $O_3(x, y) = \beta(\alpha + 1)^2x^3 - 2\beta\alpha(\alpha + 1)x^3y + \beta\alpha^2x^3y^2 - \beta\alpha(1 + 2\alpha)x^4 + 2\beta\alpha^2x^4y + \beta\alpha^2x^5 - \beta(1 - x - y)^2O_4(x)$

(2) $A(0, 0) = \begin{pmatrix} a_0 & 0 \\ b_0 & b_1 \end{pmatrix}$

Proof. For simplicity, let us consider the following system:

$$\begin{cases} \dot{x} = \dfrac{\beta x(1 - x - y)^2}{1 + \alpha x} - \gamma x := f_0(x, y), \\ \dot{y} = (\delta x - y) := g_0(x, y) \end{cases} \tag{4.2}$$

We need to show that $f_0 = f$ and $g_0 = g$. First we will change $f_0(x,y)$ into geometric series. Then (4.2) will become:

$$\begin{aligned}
\text{Let } h(x,y) &= \beta x(1-x-y)^2 \left[\sum_{n=0}^{\infty}(-1)^n \alpha^n x^n\right] \\
h(x,y) &= \frac{\beta x}{\alpha x}(1-x-y)^2 \left[1 - \frac{1}{1+\alpha x}\right] \\
&= \frac{\beta}{\alpha}(1-x-y)^2 \left[1 - (1 - \alpha x + \alpha^2 x^2 - \alpha^3 x^3 + \sum_{n=0}^{\infty}(-1)^n \alpha^n x^n)\right] \\
&= \frac{\beta}{\alpha}(1-x-y)^2 \left[\alpha x - \alpha^2 x^2 + \alpha^3 x^3 - \sum_{n=4}^{\infty}(-1)^n \alpha^n x^n\right] \\
&= \beta(1-x-y)^2 \left[x - \alpha x^2 + \alpha^2 x^3 - O_4(x)\right] \\
&= \beta(1 - 2x - 2y + 2xy + x^2 + y^2) \left[x - \alpha x^2 + \alpha^2 x^3 - O_4(x)\right] \\
&= \beta \left[x - \alpha x^2 + \alpha^2 x^3 - O_4(x)\right] \\
&+ \beta \left[-2x^2 + 2\alpha x^3 - 2\alpha^2 x^4 + 2xO_4(x)\right] \\
&+ \beta \left[-2xy + 2\alpha x^2 y - 2\alpha^2 x^3 y + 2yO_4(x)\right] \\
&+ \beta \left[2x^2 y - 2\alpha x^3 y + 2\alpha^2 x^4 y - 2xyO_4(x)\right] \\
&+ \beta \left[x^3 - \alpha x^4 + \alpha^2 x^5 - x^2 O_4(x)\right] \\
&+ \beta \left[xy^2 - \alpha x^2 y^2 + \alpha^2 x^3 y^2 - y^2 O_4(x)\right] \\
\text{where } O_4(x) &= \sum_{n=4}^{\infty}(-1)^n \alpha^{n-1} x^n
\end{aligned}$$

$$
\begin{aligned}
f_1(x,y) &= h(x,y) - \gamma x \\
&= (\beta-\gamma)x - 2\beta xy + \beta xy^2 - \beta(\alpha+2)x^2 \\
&+ \ 2\beta(\alpha+1)x^2y - \beta\alpha x^2y^2 + O_3(x,y) \\
\text{where } O_3(x,y) &= \beta(\alpha+1)^2x^3 - 2\beta\alpha(\alpha+1)x^3y + \beta\alpha^2x^3y^2 - \beta\alpha(1+2\alpha)x^4 \\
&+ \ 2\beta\alpha^2x^4y + \beta\alpha^2x^5 - \beta(1-x-y)^2O_4(x). \text{ Therefore} \\
f_1(x,y) &= a_0x + a_1xy + a_2xy^2 + a_3x^2 + a_4x^2y + a_5x^2y^2 + O_3(x,y), \\
\text{and } g_1(x,y) &= -y + \delta x \\
&= b_0y + b_1x
\end{aligned}
$$

Then the result follows for Lemma 4.2.2**(1)**.

For the result of Lemma 4.2.2**(2)**, we see that (2.2), the Jacobian matrix $A(0,0)$ of (3.5) is equivalent to the Jacobian matrix of $A(0,0)$ of (4.2). Thus,

$$A(0,0) = \frac{\partial(f,g)}{\partial(x,y)}|_{x=y=0} = \begin{pmatrix} a_0 & 0 \\ b_0 & b_1 \end{pmatrix}$$

□

Now, the result follows for the dynamics of (3.5) at the equilibria

Theorem 4.2.3. *Let β, γ, α and $\delta > 0$, If $\beta = \gamma$ then $(0,0)$ is a saddle-node of* (3.5)

Proof. We use Lemma 2.1.6 to prove Theorem 4.2.3. We know that $(0, 0)$ is an equilibrium of (3.5) and $\beta = \gamma$. By computation, $a_0 = 0$ and $a_3 = -\beta(2+\alpha)$. By Lemma 4.2.2 **(1)**, the equation of (3.5) becomes:

$$\begin{cases} \dot{x} = -2\beta xy + \beta xy^2 - \beta(2+\alpha)x^2 + 2\beta(\alpha+1)x^2y - \beta\alpha x^2y^2 + O_3(x,y) \\ \dot{y} = -y + \delta x \end{cases} \tag{4.3}$$

Let $y_1 = y - \delta x$ and $x_1 = x$, this implies $y_1 + \delta x_1 = y$ and $x = x_1$

$$\begin{aligned}
\text{Then } \dot{y_1} &= \dot{y} - \delta\dot{x} \\
&= (-y + \delta x) - \delta\left(-2\beta xy + \beta xy^2 - \beta(2+\alpha)x^2\right) \\
&- \delta\left(2\beta(\alpha+1)x^2y - \beta\alpha x^2y^2 + O_3(x,y)\right) \\
&= ((-y_1 - \delta x_1) + \delta x_1) - \delta\left(-2\beta x_1(y_1 + \delta x_1) + \beta x_1(y_1^2 + 2\delta x_1 y_1 + \delta^2 x_1^2)\right) \\
&- \delta\left(-\beta(\alpha+2)x_1^2 + 2\beta(\alpha+1)x_1^2(y_1 + \delta x_1)\right. \\
&- \delta\left(-\beta\alpha x_1^2(y_1^2 + 2\delta x_1 y_1 + \delta^2 x_1^2) + O_3(x_1,y_1)\right) \\
&= -y_1 - \delta\left(-2\beta x_1 y_1 - 2\delta\beta x_1^2 + \beta x_1 y_1^2 + 2\delta\beta x_1^2 y_1 + \delta^2\beta x_1^3\right) \\
&- \delta\left(-\beta(\alpha+2)x_1^2 + 2\beta(\alpha+1)x_1^2 y_1 + 2\delta\beta(\alpha+1)x_1^3 - \beta\alpha x_1^2 y_1^2\right) \\
&- \delta\left(-2\delta\beta x_1^3 y_1 - \delta^2\beta x_1^4 + O_3(x_1, y_1 + \delta x_1)\right) \\
&= -y + 2\delta\beta x_1 y_1 - \delta\beta x_1 y_1^2 + \delta\beta(2\delta + \alpha + 2)x_1^2 - 2\delta\beta(\delta + 2(\alpha+))x_1^2 y_1 \\
&+ \delta\beta\alpha x_1^2 y_1^2 + P_3(x_1, y_1) := \varrho y_1 + q(x_1, y_1),
\end{aligned}$$

where $P_3(x_1, y_1) = -\delta^2\beta(\delta+2(\alpha+1))x_1^3+2\delta^2\beta x_1^3 y_1-\delta^3\beta\alpha x_1^4+\delta O_3(x_1, y_1+\delta x_1)$ begins with third order terms.

$$\begin{aligned}
\text{Let } \dot{x_1} &= \dot{x} = -2\beta xy + \beta xy^2 - \beta(2+\alpha)x^2 + 2\beta(\alpha+1)x^2y - \beta\alpha x^2y^2 + O_3(x,y) \\
&= -2\beta x_1(y_1+\delta x_1) + \beta x_1(y_1^2 + 2\delta x_1 y_1 + \delta^2 x_1^2) - \beta(\alpha+2)x_1^2 \\
&+ 2\beta(\alpha+1)x_1^2(y_1+\delta x_1) - \beta\alpha x_1^2(y_1^2 + 2\delta x_1 y_1 + \delta^2 x_1^2) + O_3(x_1, y_1+\delta x_1) \\
&= -2\beta x_1 y_1 - 2\delta\beta x_1^2 + \beta x_1 y_1^2 + 2\delta\beta x_1^2 y_1 + \delta^2\beta x_1^3 - \beta(\alpha+2)x_1^2 - \delta^2\beta\alpha x_1^4 \\
&+ 2\beta(\alpha+1)x_1^2 y_1 + 2\delta\beta x_1^3 - \beta\alpha x_1^2 y_1^2 - 2\delta\beta\alpha x_1^3 y_1 + O_3(x_1, y_1+\delta x_1) \\
&= -2\beta x_1 y_1 + \beta x_1 y_1^2 - \beta(2\delta+\alpha+2)x_1^2 + 2\beta(\alpha+\delta+1)x_1^2 y_1 \\
&- \beta\alpha x_1^2 y_1^2 + H_3(x_1, y1) := p(x_1, y_1),
\end{aligned}$$

where $H_3(x_1, y_1) = \delta\beta(\delta+2(\alpha+1))x_1^3 - 2\delta\beta\alpha x_1^3 y_1 - \delta^2\beta\alpha x_1^4 + O_3(x_1, y_1)$ begins with third order terms.

Since $\varrho = -1 < 0$ and $a_{20} = -\beta(2(\delta+1)+\alpha)$ which $a_{20} < 0$ and $b_{20} = \delta\beta(2(\delta+1)+\alpha) > 0$. It follows from Lemma 2.1.6 and (2.3) that $(0,0)$ is a saddle-node of (3.5) $\square$

The Figures 4.1, 4.2 and 4.3 below illustrate the phase portrait for Theorem 4.2.1 with the description of the evolution of the trajectories .

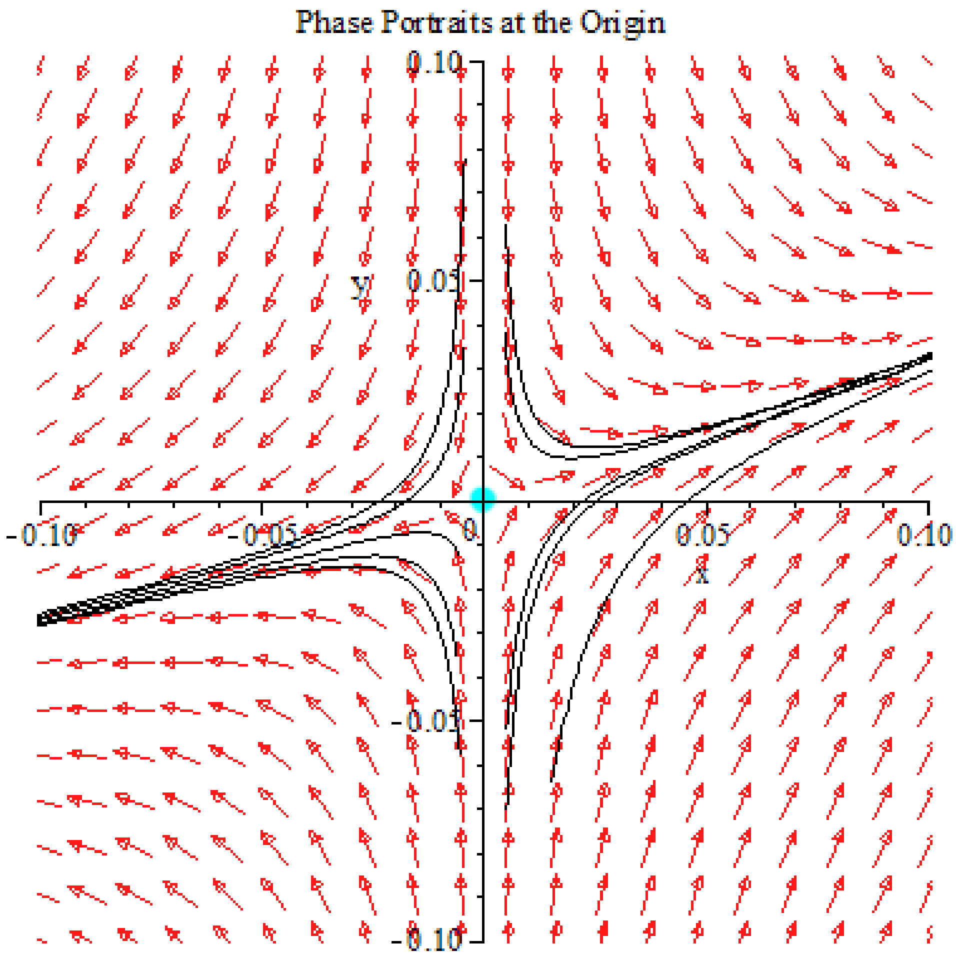


Figure 4.1: Illustrates the results of Theorem 4.2.1(1)-Saddle point by letting $\delta = \alpha = 0.5$ and by letting $\gamma = 0.1 < \beta = 0.8$. The trajectories start out at $(0, 0)$, then move away to infinitely distant as $t- > \infty$. Thus it is a saddle point at $(0, 0)$. In biological meaning, the disease can persist and cause an outbreak.

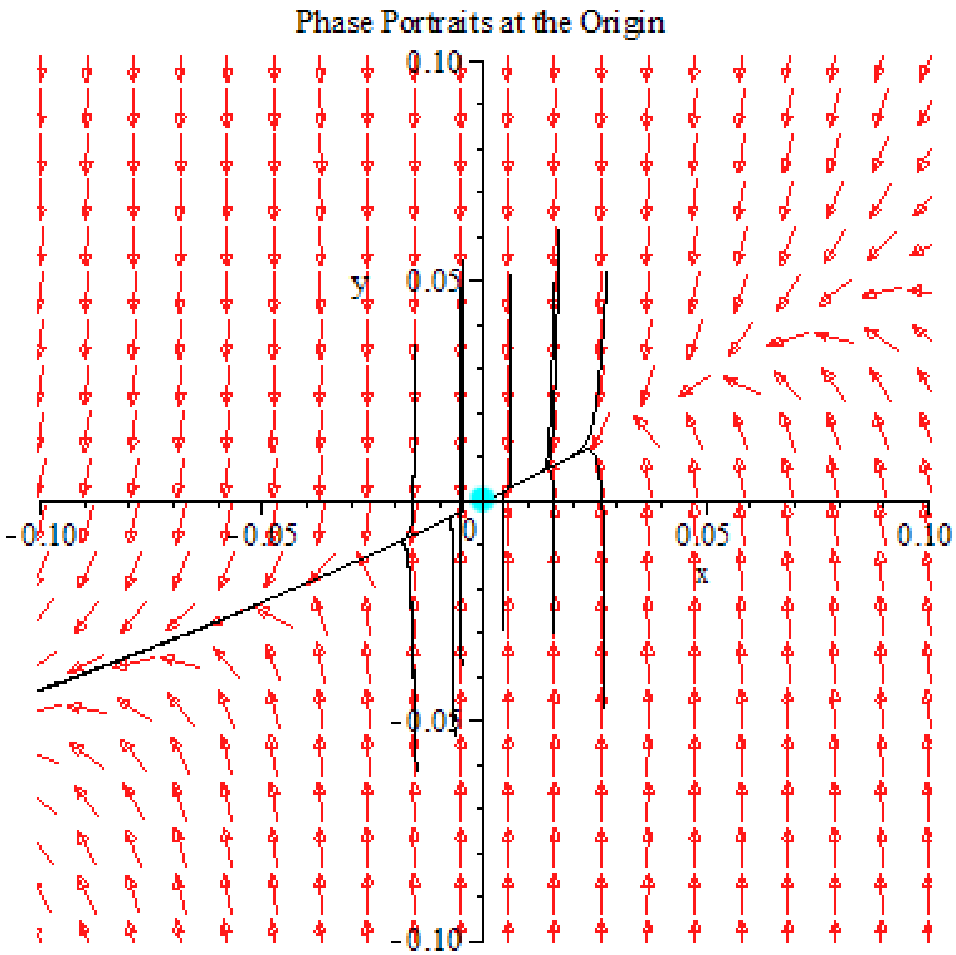


Figure 4.2: Illustrates the results of Theorem 4.2.1(2)-Saddle node by letting $\delta = \alpha = 0.5$ and by letting $\beta = \gamma = 0.5$. Since $(0, 0)$ is an isolated equilibrium point and it starts at the equilibrium, the trajectories will move toward the equilibrium but it will not converge to $(0, 0)$ as $t- > \infty$. The disease can persist at the isolated point.

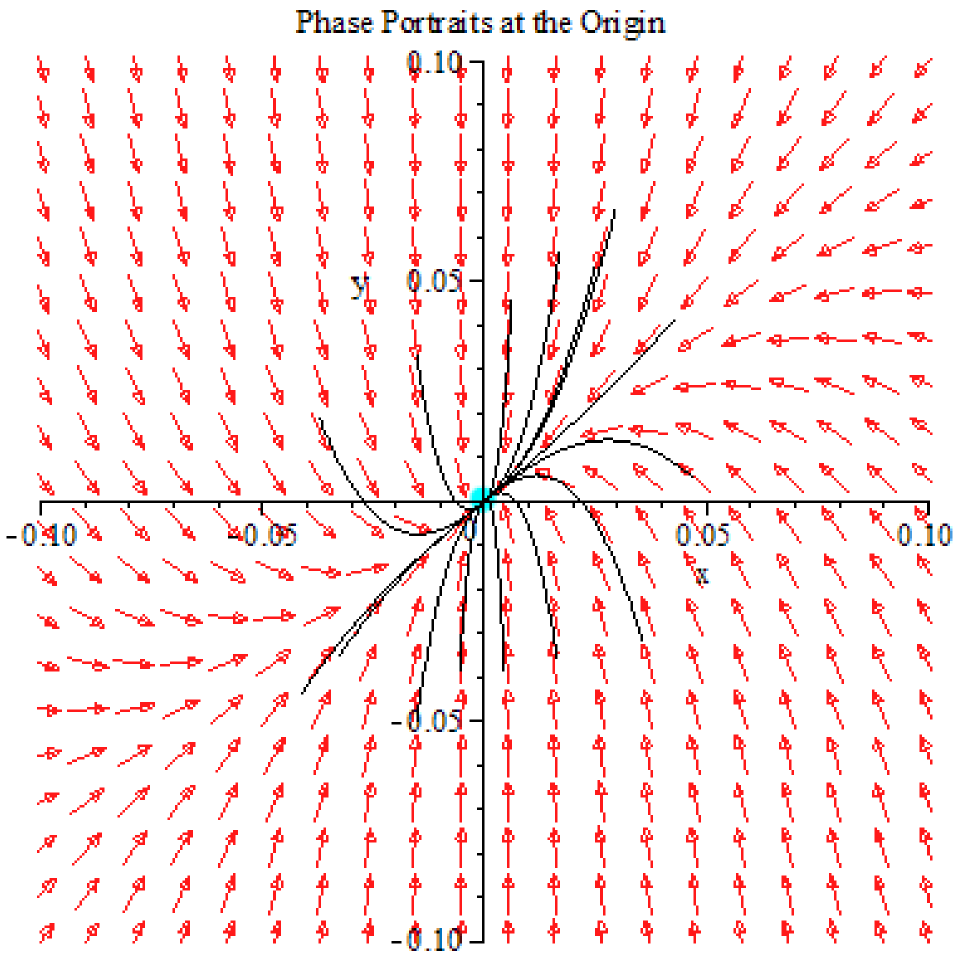


Figure 4.3: Illustrates the results of Theorem 4.2.1(3)- Stable node by letting $\delta = \alpha = 0.5$ and by letting $\beta = 0.1 < \gamma = 0.5$. The trajectories move toward the critical point $(0, 0)$ from infinite distant and then converge to the critical point as $t- > \infty$. In biological interpretation, there is no risk of outbreak of the disease since the recovery and susceptible will go to zero.

4.3 Phase portraits near (x^*, y^*)

In this section, we study the phase portraits for (x_1, y_1) and (x_2, y_2) by using Lemma 2.1.4 and 2.1.5. Before we wish to analyse the system, we need to find the determinant and the trace of (3.5).

4.3.1 Determinant and Trace for (x^*, y^*)

First, we need to derive the formulas for the determinants and traces of the Jacobian matrix $A(x^*, y^*)$ of f and g given in (3.5) at the equilibria (x^*, y^*) satisfying (3.6). By (2.2), we have

$$A(x^*, y^*) = \begin{pmatrix} \left[\dfrac{2\beta x^*(x^*+y^*-1)}{(1+\alpha x^*)} + \dfrac{\beta(1-x^*-y^*)^2}{(1+\alpha x^*)^2}\right] - \gamma & \dfrac{\beta x^*(x^*+y^*-1)}{(1+\alpha x^*)} \\ \delta & 1 \end{pmatrix} \tag{4.4}$$

By (3.6), we have $\dfrac{\beta(1-x^*-y^*)^2}{(1+\alpha x^*)} = \gamma$. This together with (4.4) give the following results formulas for $|A(x^*, y^*)|$ and $tr(A(x^*, y^*))$

Lemma 4.3.1. *Let* $\gamma(\beta, \delta) = \dfrac{1+\delta+2\beta}{2}$ *and* $\beta(\alpha, \delta, \gamma) = \dfrac{\alpha(1+\delta)(1+\gamma) - 2\alpha\gamma}{2(1+\delta)}$ *Assume that* (x^*, y^*) *satisfies* (3.6). *Then the following formulas hold.*

(1) $|A(x^*, y^*)| = \dfrac{-1}{1+\alpha x^*}\left[(2\beta(1+\delta)+\alpha\gamma)x^* - 2(\beta-\gamma)\right]$

(2) $tr(A(x^*, y^*)) = \dfrac{1}{(1+\alpha x^*)(1+\delta)}\left[2(1+\delta)(\beta - \beta(\alpha, \delta, \gamma))x^* + 2(\gamma - \gamma(\beta, \delta))\right]$

Proof. By (4.4), $\dfrac{\beta(1-x^*-y^*)^2}{(1+\alpha x^*)} = \gamma$ and $x^{*2} = \dfrac{2\beta(1+\delta)+\alpha\gamma}{\beta(1+\delta)^2}x^* - \dfrac{(\beta-\gamma)}{\beta(1+\delta)^2}$, then the determinant of $A(x^*, y^*)$:

$$
\begin{aligned}
|A(x^*, y^*)| &= \left(\frac{-2\beta x^*(1-x^*-y^*)-\gamma\alpha x^*}{1+\alpha x^*}\right)(-1) - \delta\left(\frac{-2\beta x(1-x-y)}{1+\alpha x}\right) \\
&= \frac{1}{1+\alpha x^*}\left(2\beta x^*(1-x^*-y^*)+\gamma\alpha x^* + 2\beta\delta x^*(1-x^*-y^*)\right) \\
&= \frac{1}{1+\alpha x^*}\left([2\beta x^*(1-x^*(1+\delta))](1+\delta)+\gamma\alpha x^*\right) \\
&= \frac{1}{1+\alpha x^*}\big((2\beta x^* - 2\beta x^{*2})(1+\delta))(1+\delta)+\alpha\gamma x^*\big) \\
&= \frac{1}{1+\alpha x^*}\big(-2\beta x^{*2}(1+\delta)^2 + 2\beta x^*(1+\delta)+\alpha\gamma x^*\big) \\
&= \frac{1}{1+\alpha x^*}\left(-2\beta(1+\delta)^2\left[\frac{2\beta(1+\delta)+\alpha\gamma}{\beta(1+\delta)^2}x^* - \frac{(\beta-\gamma)}{\beta(1+\delta)^2}\right]\right) \\
&+ \frac{1}{1+\alpha x^*}\left(2\beta x^*(1+\delta)+\alpha\gamma x^*\right) \\
&= \frac{1}{1+\alpha x^*}(-4\beta(1+\delta)x^* - 2\alpha\gamma x^* + 2\beta(1+\delta)x^* + 2(\beta-\gamma)+\alpha\gamma x^*) \\
&= \frac{-1}{1+\alpha x^*}[(2\beta(1+\delta)+\alpha\gamma)x^* - 2(\beta-\gamma)] =: \frac{-D(x^*)}{1+\alpha x^*}
\end{aligned}
$$

By computation, we obtain the trace of (x^*, y^*)

$$
\begin{aligned}
\rho &= \frac{-2\beta x^*(1-x^*-y^*) - \alpha\gamma x^*}{(1+\alpha x^*)} - 1 \\
&= \frac{1}{(1+\alpha x^*)}[-2\beta x^*(1-x^*(1+\delta)) - \alpha\gamma x^* - \alpha x^* - 1] \\
&= \frac{1}{(1+\alpha x^*)}[-2\beta x^* + 2\beta(1+\delta)x^{*2} - \alpha\gamma x^* - \alpha x^* - 1] \\
&= \frac{1}{(1+\alpha x^*)}\left[-2\beta x^* + 2\beta(1+\delta)\left[\frac{2\beta(1+\delta)+\alpha\gamma}{\beta(1+\delta)^2}x^* - \frac{(\beta-\gamma)}{\beta(1+\delta)^2}\right]\right] \\
&- \frac{1}{(1+\alpha x^*)(1+\delta)}\alpha\gamma x^* - \alpha x^* - 1 \\
&= \frac{1}{(1+\alpha x^*)}[-2\beta(1+\delta)x^* + 4\beta(1+\delta)x^* + 2\alpha\gamma x^* - 2(\beta-\gamma)] \\
&- \frac{1}{(1+\alpha x^*)(1+\delta)}(1+\delta)(\alpha x^* + \alpha\gamma x^* + 1) \\
&= \frac{1}{(1+\alpha x^*)(1+\delta)}[2\beta(1+\delta)x^* + 2\alpha\gamma x^* - \alpha(1+\delta)(1+\gamma)x^*] \\
&- \frac{1}{(1+\alpha x^*)(1+\delta)}(1+\delta) - 2(\beta-\gamma) \\
&= \frac{1}{(1+\alpha x^*)(1+\delta)}[(2\beta(1+\delta) + 2\alpha\gamma - \alpha(1+\delta)(1+\gamma))x^*] \\
&+ \frac{1}{(1+\alpha x^*)(1+\delta)}2\gamma - 1 - \delta - 2\beta \\
&= \frac{1}{(1+\alpha x^*)(1+\delta)}\left[(2(1+\delta)(\beta - \beta(\alpha,\delta,\gamma))x^* + 2(\gamma - \gamma(\beta,\delta))\right]
\end{aligned}
$$

$\square$

4.3.2 Dynamical properties of (x_1, y_1) and (x_2, y_2)

The following results gives the dynamical properties of (3.5) near (x_1, y_1) and (x_2, y_2).

Theorem 4.3.2. *Assume that one of the following conditions holds:*
(a) If $\alpha, \delta, \beta > 0$ *and* $\gamma > 0$*, then* (x_2, y_2) *is a saddle of* (3.5).
(b) If $\alpha, \delta, \beta, \gamma > 0$ *and* $0 < \gamma < \beta$ *and* $\beta \leq \beta(\alpha, \delta, \gamma)$*, then* (x_1, y_1) *is a stable node or focus of* (3.5).

Proof. By Lemma 4.3.1 (1), we have for (x_2, y_2),

$$\begin{aligned}
|A(x_2, y_2) &= \frac{-D(x_2)}{1 + \alpha x_2} \\
-D(x_2) &= -(2\beta(1+\delta) + \alpha\gamma)\left[\frac{2\beta(1+\delta) + \alpha\gamma + \sqrt{\Delta}}{2\beta(1+\delta)^2}\right] + 2(\beta - \gamma) \\
&= \frac{1}{2\beta(1+\delta)^2}\left(-4\beta^2(1+\delta)^2 - 4\beta\alpha\gamma(1+\delta) - 2\beta(1+\delta)\sqrt{\Delta}\right) \\
&+ \frac{1}{2\beta(1+\delta)^2}\left(-\alpha^2\gamma^2 - \alpha\gamma\sqrt{\Delta} + 4\beta^2(1+\delta)^2 - 4\beta\gamma(1+\delta)^2\right) \\
&= \frac{-1}{2\beta(1+\delta)^2}\left(\alpha^2\gamma^2 + 4\beta\alpha\gamma(1+\delta) + 4\beta\gamma(1+\delta)^2\right) \\
&+ \frac{-1}{2\beta(1+\delta)^2}\left(2\beta(1+\delta) + \alpha\gamma)\sqrt{\Delta}\right) < 0
\end{aligned}$$

Since $x_2 > 0$, we have $|A(x_2, y_2)| < 0$. The result follows from Lemma 2.1.5 (i) and Theorem 2.1.4(ii).

By Lemma 4.3.1 (1), we have for (x_1, y_1),

$$\begin{aligned}
|A(x_1,y_1) &= \frac{-D(x_1)}{1+\alpha x_1} \\
-D(x_1) &= -\left(2\beta(1+\delta)+\alpha\gamma\right)\left[\frac{2\beta(1+\delta)+\alpha\gamma-\sqrt{\Delta}}{2\beta(1+\delta)^2}\right]+2(\beta-\gamma) \\
&= \frac{1}{2\beta(1+\delta)^2}\left(-4\beta^2(1+\delta)^2-4\beta\alpha\gamma(1+\delta)+2\beta(1+\delta)\sqrt{\Delta}\right) \\
&+ \frac{1}{2\beta(1+\delta)^2}(-\alpha^2\gamma^2+\alpha\gamma\sqrt{\Delta}+4\beta^2(1+\delta)^2-4\beta\gamma(1+\delta)^2) \\
&= \frac{-1}{2\beta(1+\delta)^2}\left(\alpha^2\gamma^2+4\beta\alpha\gamma(1+\delta)+4\beta\gamma(1+\delta)^2\right) \\
&+ \frac{1}{2\beta(1+\delta)^2}\left((2\beta(1+\delta)+\alpha\gamma)\sqrt{\Delta}\right) \\
&= \frac{-1}{2\beta(1+\delta)^2}\left(\Delta-(2\beta(1+\delta)+\alpha\gamma)\sqrt{\Delta}\right) \\
&= \frac{-\sqrt{\Delta}}{2\beta(1+\delta)^2}\left(\sqrt{\Delta}-(2\beta(1+\delta)+\alpha\gamma\right)
\end{aligned}$$

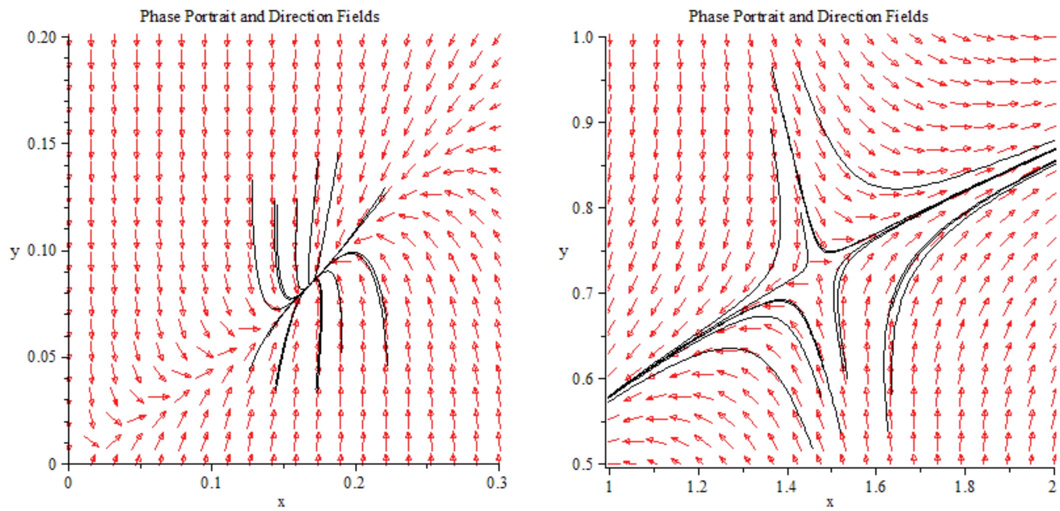
From Lemma 3.5.1,we know that $2\beta(1+\delta)+\alpha\gamma > \sqrt{\Delta}$ if and only if $\gamma < \beta$, then $|A(x_2, y_2)| > 0$. From Lemma 4.3.1 (2), we have for (x_1, y_1)

$$tr(A(x_1,y_1)) = \frac{1}{(1+\alpha x_1)(1+\delta)}\left[2(1+\delta)(\beta-\beta(\alpha,\delta,\gamma))x_1+2(\gamma-\gamma(\beta,\delta))\right]$$

Furthermore, we restrict $\gamma < \beta$, now we are going to show that $\beta < \gamma(\beta, \delta)$

$$\beta-\gamma(\beta,\delta) = \beta-\left[\frac{2\beta+\delta+1}{2}\right] = \frac{2\beta-2\beta-\delta-1}{2} < 0.$$

Since $\gamma < \beta$ and $\beta < \gamma(\beta, \delta)$, this implies that $\gamma < \gamma(\beta, \delta)$. We already know that $x_1, \alpha, \delta > 0$ with $\beta \leq \beta(\alpha, \delta, \gamma)$ and $\gamma < \gamma(\beta, \delta)$. We can conclude that $tr(A(x_1, y_1)) < 0$. Thus from Lemma 2.1.5$(ii)(iii)$ and Theorem 2.1.4 (i) is result that (x_1, y_1) is a stable node or a stable focus of (3.5). □



(a) Stable node near the equilibrium point, (x_1, y_1)

(b) Saddle point near the equilibrium point, (x_2, y_2)

Figure 4.4: Phase portrait diagrams for Theorem 4.3.2

Remark 4.3.3. Theorem 4.3.2 provides conditions on α, β, δ and γ to ensure that (x_1, y_1) is stable, but it can not be used to validate whether (x_1, y_1) is a node or focus.

The Figure 4.4 illustrates the comparison of the saddle point and the stable node for the Theorem 4.3.2. On the left hand side, it is the phase portrait and the direction fields for the equilibrium point (x_1, y_1). While, the right hand side is corresponding to the phase portrait and the direction fields for the equilibrium point (x_2, y_2). With the help of Maple 13, when choosing $\gamma = 0.3, \delta = 0.5$ and $\alpha = 1.5$, then $\beta(\alpha, \delta, \gamma) = 0.675$. After solved (3.6), the fixed points corresponding to the parameters are: $(x_1 = 0.169052, y_1 = 0.845260)$ and $(x_2 = 1.460577, y_2 = 0.730289)$. The

Figure 4.4a indicated (x_1, y_1) indeed a stable node.

4.3.3 Determine the stability of (x_1, y_1) with the ranges of parameter

In the following result, we provide some conditions on α, β, δ and γ so that (x_1, y_1) can determine the stability of (3.5). We assume that $0 < \gamma < \beta$ and δ is very small.

Lemma 4.3.4. *Let $\alpha > 0, \gamma > 0$, then there exist $\gamma_0 > 0, \beta_0 > 0$ such that $tr(A(x_1, y_1)) < 0$ for $0 < \beta < \beta_0$ and $0 < \gamma < \gamma_0$.*

Proof.

$$\begin{aligned} \text{Let } \beta(\alpha,\gamma,\delta) &= \frac{\alpha(1+\delta)(1+\gamma) - 2\alpha\gamma}{2(1+\delta)} > 0 \\ \beta(\alpha,\gamma,0) &= \frac{\alpha(1+\gamma) - 2\alpha\gamma}{2} = \frac{\alpha(1-\gamma)}{2} > 0 \end{aligned}$$

then $\beta \in (0, \beta_0)$, where $\beta_0 = \dfrac{\alpha(1-\gamma)}{2}$ and $\gamma \in (0, \gamma_0)$, where $\gamma_0 \in (0, 1)$. Since $x_1 > 0$ and $\beta \leq \beta_0$, base on Lemma (2) the $tr(A(x_1, y_1)) < 0$.

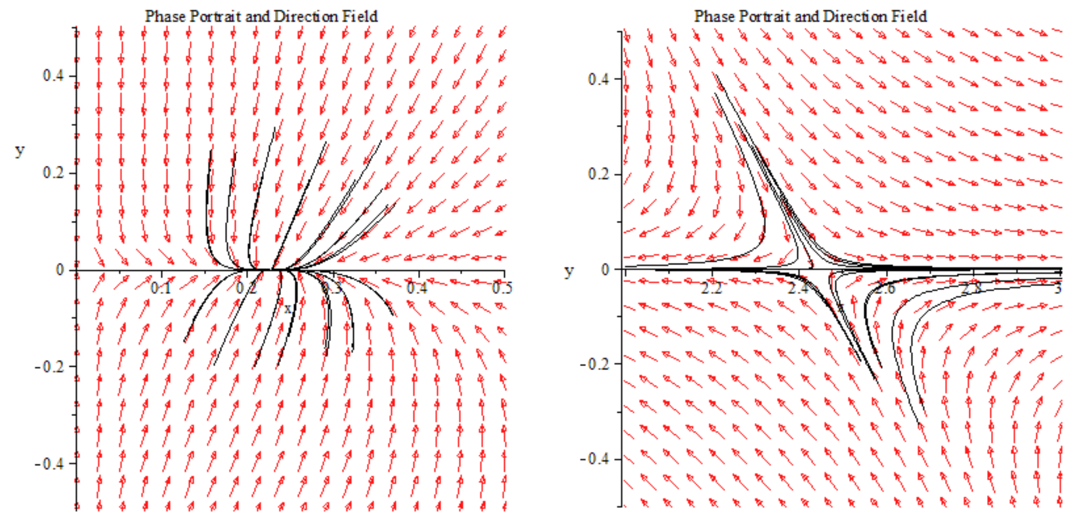
$\square$

Theorem 4.3.5. *If $\delta = 0, ,\ 0 < \beta \leq \beta_0$ and $0 < \gamma < \gamma_0$ for $\gamma_0 > 0, \beta_0 > 0$ then (x_1, y_1) is a stable node or focus of (3.5).*

Proof. When $\delta \to 0$, then $\Delta = \alpha^2\gamma^2 + 4\beta\gamma + 4\beta\alpha\gamma$ and $x_1 = \dfrac{2\beta + \alpha\gamma - \sqrt{\Delta}}{2\beta}$

This implies that the $|A(x_1, y_1)| = -\dfrac{\sqrt{\Delta}}{2\beta}\left(\sqrt{\Delta} - (2\beta + \alpha\gamma)\right)$.

Based on Lemma 3.5.1, we know that $2\beta + \alpha\gamma > \sqrt{\Delta}$ if and only if $\gamma < \beta$, thus $|A(x_1, y_1)| > 0$. We know that from Lemma 4.3.4, the $tr(A(x_1, y_1)) < 0$. The result follows. $\square$



(a) Stable node near the equilibrium point, (x_1, y_1)

(b) Saddle point near the equilibrium point, (x_2, y_2)

Figure 4.5: Phase portrait diagram for Theorem 4.3.5

The Figure 4.5 illustrate the stable node for the Theorem 4.3.5. On the left hand side, it is the phase portrait and the direction fields for the equilibrium point (x_1, y_1). While, the right hand side is corresponding to the phase portrait and the direction fields for the equilibrium point (x_2, y_2). With the help of Maple 13, we chose $\gamma = 0.3, \delta = 0, \alpha = 1.5$, and $\beta = 0.675$. After solve (3.6), the equilibria are corresponding to the parameters are: $(x_1 = 0.22779, y_1 = 0)$ and $(x_2 = 2.43887, y_2 = 0)$.

Remark 4.3.6. Since $\gamma_0 \in (0, 1)$ and $0 < \gamma < \gamma_0$, this implies that $0 < \gamma < 1$. It is equivalent to $0 < \dfrac{d+r}{d+\nu} < 1$. Theorem (4.3.5) implies that the equilibrium is stable if the rate of removed individual who lose immunity v is stronger than the recovery rate of the infective individuals ν. The lose immunity rate has a positive effect on the stability of (x_1, y_1), while the recovery rate has the negative effect. The result is similar with [24].

Chapter 5

Conclusion

In this thesis, we have used the theorems and techniques in Chapter 2 to find the number of equilibria for our SIRS model. As long as β, δ, α, and $\gamma > 0$ then our equilibria will be positive. The number of equilibria depending on the parameters β and γ. If $\beta \leq \gamma$, we have two equilibria, $(0, 0)$, and (x_2, y_2). Vice versa, if $\gamma < \beta$, then we have three equilibria $(0, 0), (x_1, y_1)$ and (x_2, y_2). Further analysis of the stability of each equilibrium depends on the restriction of the parameters β and γ.

We have shown that at the disease-free equilibrium , the three possible cases can happen: saddle , saddle node and stable node . The biological interpretation for the disease-free equilibrium for each stage as follow:

1. saddle - the disease can still persist and cause an outbreak.

2. saddle node - the disease can still persist at the isolated point.

3. stable node - there is no risk of outbreak of the disease since the infective

and recovery will go to zero.

The first case is similar to [24], the equilibrium $(0, 0)$ is unstable and disease-free, that is, there are some solutions $(I(t), R(t))$, where $I(t)$ may not converge to 0. For second and third cases are new. We can clarify the parameters α, β, γ and δ that the solution will converge to 0.

After studying the dynamical behaviours of (3.5) near its interior equilibria, (x_1, y_1) and (x_2, y_2), we can conclude the stability of our equilibria. Regardless $\beta < \gamma$ and vice versa, (x_2, y_2) is always the saddle of (3.5). We can see that any solution $(x(t), y(t))$ of (3.5) can not converge to (x_2, y_2) as $t- > \infty$. Then there will be an outbreak of the disease. Thus, the infective and the recovery individual cannot be controlled.

After the restriction of the parameters, especially $\gamma < \beta$. We can see that (x_1, y_1) is a stable node or focus of (3.5). Regardless where we start the solution, it will converge to (x_1, y_1) as $t- > \infty$. Therefore, both the infective and the recovery near (x_1, y_1) can co-exist under the condition Theorem 4.3.2. The biological meaning that the disease can not be eradicated, it can not spread out and can be controlled at a number near x_1.

5.1 Limitations

As mentioned in our introduction, our SIRS model is a nonlinear differential equation and we can only study the local stability near the equilibrium after we linearize the model. To analyze the global stability near the equilibrium, we need further research, which is out of the scope of this study.

Another limitation to our model is that we can not validate whether (x_1, y_1) is a node or focus even though we restrict the parameters for α, β, γ and let δ approach to 0. But with the help of Maple, we can use a graphical approach to verify the stability of (x_1, y_1)

5.2 Future study

For future study, it is worthwhile to explore the global stability near the equilibrium by using Dulac criterion and Poincar$\acute{e}$ -Bendixon theorem and by constructing a Lyapunov function (see [26] for more detail for constructing Lyapunov function for SIR and SIRS model). With our SIRS model, we can replace the removed individual $R(t)$ in the first equation of (3.1) and study the reduced SI system and compare with our reduced IR

Our epidemic model is a deterministic model, and the biological systems are subject to random fluctuation. Moreover, Pathak and Maiti has stated most natural phenomena do not follow deterministic laws, but rather oscillate randomly about some average that the deterministic equilibrium is not an absolutely fixed state; instead it is a "fuzzy". For more detail, see [2, 3]. Therefore, in deterministic environment, it is hard to predict the future outcome. A stochastic environment should be considered for our model in future work.

Bibliography

[1] A Major Outbreak of Severe Acute Respiratory Syndrome in Hong Kong, *The New England Journal of Medicine*, `http://www.nejm.org/doi/full/10.1056/NEJMoa030685#t=article`, **Accessed: 10/06/2012 (2012).**

[2] A.Lahrouz, L.Omari and D.Kiouach, Global analysis of a deterministic and stochastic nonlinear SIRS epidemic model, *Nonlinear Analysis: Modelling and Control*, **16 (2011), 57-76.**

[3] A. Maiti and S.Pathak, A Modified Holling-Tanner Model in Stochastic Environment, *Nonlinear Analysis: Modelling and Control*, **14 (2009), 51-71.**

[4] **C.Lizarrage-Celaya, Maple and Mathematica,** *A Problem Solving Approach for Mathematics*, **Springer (2007).**

[5] **C. Neuhauser, Differential Equation,** *Calculus for Biology and Medicine 3rd*, **8 (2011), 390-425.**

[6] **C. Neuhauser, System of Differential Equations,** *Calculus for Biology and Medicine 3rd*, **11 (2011), 587-646.**

[7] **C.R. Zhu and K. Q. Lan, Phase portraits, Hopf bifurcations and limit cycles of Leslie-Gower predator-prey systems with harvesting rates,** *Discrete Contin. Dyn. Syst. Ser. B*, **14 (2010), 289-306.**

[8] **C.R. Zhu and K.Q.Lan Phase portraits, Hopf Bifurcations and limit cycles of the Holling-Tanner models for predator-prey interactions,** *Nonlinear Analysis: Real World Applications*, **12 (2011), 1961-1973.**

[9] **C.R. Zhu and K.Q.Lan, Phase portraits of predator - prey systems with harvesting rates,** *Discrete and Continuous Dynamical Systems*, **32(3) (2012), 901-933.**

[10] **C.Sun and W.Yang, Global results for an SIRS model with vaccination and isolation,** *Nonlinear Analysis: Real World Application*, **11 (2010), 4223-4237.**

[11] **D.Gonze and M.kaufman, Theory of non-linear dynamical systems,** `homepages.ulb.ac.be/~dgonze/TEACHING/nonlinear.pdf`, **Accessed: 08/06/2012 (2011).**

[12] **F.Brauer and C.Castillo-Chavez, Mathematical Models in Population Biology and Epidemiology,** *Springer*,**4-9 (2010), 125-165, 345-409.**

[13] **H.W Hethcote, H.W Stech and P. van den Driessche, Stability analysis for models of diseases without immunity,** *J. Math. Biol.*, **29 (1981), 271-287.**

[14] **H.W Hethcote, H.W Stech and P. van den Driessche, Some epidemiological models with nonlinear incidence,** *J. Math. Biol.*, **29 (1991), 271-287.**

[15] **J.Wang and Y.Xue, Bifurcation analysis of a stage-structured epidemic model with a nonlinear incidence,** *International Journal of Information and Systems Sciences*, **7, 1 (2011) 61-72.**

[16] **Linda J.S. Allen, Nonlinear Ordinary Differential Equations: Theory and Examples,** *An Introduction To Mathematical Biology*, **5 (2007), 176-232.**

[17] **L.Wen and X.Yang, Global stability of a delayed SIRS model with temporary immunity,** *Chaos Solitons and Fractals*, **38 (2008), 221-226.**

[18] **M.E. Alexander and S.M. Moghadas, Bifurcation analysis of an SIRS epidemic model with generalized incidence,** *SIAM J. Appl. Math*, **65 (2005), 1794-1816.**

[19] **PubMed Health,** *U.S. National Library Of Medicine*, `www.ncbi.nlm.gov/pubmedhealth`, **Accessed: 10/06/2012 (2012).**

[20] **R.Sun, Global Stability of the endemic equilibrium of multi-group SIR models with nonlinear incidence,** *Computers and Mathematics with Applications*,**60 (2010), 2286-2291.**

[21] **Seasonal Influenza (Flu),** *Centers for Disease Control and Prevention*, `www.cdc.gov/ncidod/diseases/flu/fluvirus.htm`, **Accessed: 10/06/2012 (2012).**

[22] **S. Haensch et Al, Distinct Clones of Yersinia pestis Caused the Black Death,** *PLoS Pathog*, **6 (10) (2009) , e1001134**

[23] **W.Liu, S.Levin and Y.Iwasa, Influence of nonlinear incidence rate upon the behavior of SIRS epidemiological models,** *Journal of Mathematical Biology*, **13 (1986), 185-198.**

[24] **S.Ruan and W.Wang, Dynamical behavior of an epidemic model with a nonlinear incidence rate,** *Journal of Differential Equation*,**188 (2002), 135-163.**

[25] **V. Capasso and G. Serio, A generalization of the Kermack-McKendrick deterministic epidemic model,** *Math. Biosci.*, **42 (1978), 43-61.**

[26] **V.Pokrovskii et al, Lyapunov functions for SIR and SIRS epidemic model.** *Applied Mathematics Letters*, **23 (2010), 446-448.**

[27] **Z. Song, J. Xu and Q. Li, Local and global bifurcations in an SIRS epidemic model,** *Appl. Math. Comput.*, **214 (2009), 534-547.**

Appendix A

Maple Coding

A.1 Code for ploting Nullclines

With the help of Maple13, the code for the Nullclines of the SIRS model as follow:

$with(plots):$

$l1 := plot(0.8*(1-x*(1+0.5))^2-0.5*(1+0.5*x), x = 0..3, color = cryan):$

$l2 := plot(0.5*x, x = 0..3, linestyle = [longdash]):$

$l3 := plot(0.3*(1-x*(1+0.5))^2-0.8*(1+0.5*x), x = 0..3, color = green):$

$l4 := plot(0.8*(1-x*(1+0.5))^2-0.8*(1+0.5*x), x = 0..3, color = blue):$

$display(l1, l2, l3, l4, view = [0..3, -2..2], labels = [x-Infective, y-Recovery], thickness = 2,]$ title='The Nullclines of Infective and Recovery');

A.2 Code for solving (3.6)

$with(linalg)$:

$f := \frac{beta.x.(1-x-y)^2}{1+alpha.x} - gamma1.x = 0;$

$g := delta.x - y = 0;$

$beta := 0.675;$

$gamma1 := 0.3;$

$delta := 0.5;$

$alpha := 1.5;$

$solve(\{f, g\}, \{x, y\});$

A.3 Code for phase portrait

The code for the phase portrait for (x_1, y_1) and (x_1, y_1)

$restart:$

$with(DEtools), with(linalg):$

$beta := 0.675$

$gamma1 := 0.3$

$alpha := 1.5$

$delta := 0.5$

$sys := \{diff(x(t), t) = \dfrac{beta.x(t).(1 - x(t) - y(t))^2}{1 + alpha.x(t)} - gamma1.x(t), diff(y(t), t) = delta.x(t) - y(t)\};$

$DEplot(sys, [x(t), y(t)], t = 0..50, x = 0..2, y = 0..2, [[x(0) = 0, y(0) =$

$0]], stepsize = 0.1, linecolor = blue, thickness = 1, arrows = slim);$

Index